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Chenciner, Alain (F-ODP-IMC)
Poincaré and the three-body problem. (English summary)
Henri Poincaré, 1912–2012, 51–149, *Prog. Math. Phys.*, 67, *Birkhäuser/Springer, Basel*, 2015.

Bernard of Chartres is frequently cited for saying that we see more and farther than our predecessors, not because of the acuteness of our sight or the stature of our body, but because we are carried aloft and elevated by the magnitude of the giants. Illustrating the spirit of this metaphor, Chenciner provides a unique insight into how the modern theory of dynamical systems has been erected atop Henri Poincaré's work on celestial mechanics, particularly on the Three-Body Problem. Furthermore, Chenciner argues that many of the current ideas in the field have emerged, in one way or another, from Poincaré's investigations.

Chenciner's writing defies usual genre classification. While it encompasses elements of an *exposition* on the Three-Body Problem, a *review* of Poincaré's work, and a *survey* of its ramifications throughout the field of dynamical systems, it goes well beyond these, by providing fresh perspectives, unraveling subtle connections, and asserting personal viewpoints. To best capture the original character of his writing, I will refer to it as a *mathematical essay*. The style of the writing is literary. Many of the illustrations are beautiful hand drawings of incredible detail that can easily qualify as works of art. Abundant original citations from Poincaré (in French, with English footnote translations) are given and carefully commented upon. A section titled 'Regret' laments that the beautiful language of Poincaré is less and less 'audible'.

The essay is focused on *Les méthodes nouvelles de la mécanique céleste*, whose three volumes, totaling about 1300 pages, appeared in 1892, 1893, and 1899, respectively [H. Poincaré, *Tome I*, reprint of the 1892 original, Grands Class. Gauthier-Villars, Lib. Sci. Tech. Albert Blanchard, Paris, 1987; MR0926906; *Tome II*, reprint of the 1893 original; MR0926907; *Tome III*, reprint of the 1899 original; MR0926908]. It is worth mentioning that the reading of the three volumes of *Méthodes nouvelles* made the subject of a three-year long seminar led by Chenciner and Laskar at the Bureau des Longitudes (of which Poincaré himself used to be a member as well as president), which was attended by astronomers and mathematicians, between 1988 and 1990.

The diegesis of the essay relies on the intimate connections between analytic computations of the orbits of the celestial bodies and the geometric objects underlying these computations.

After Section 1 (Introduction), Section 2 begins by discussing the equations of the N-body problem, the Kepler problem, and the planetary problem viewed as a perturbation of $(N-1)$ fictitious Kepler problems. This latter model is used to motivate what Poincaré described as the "general problem of dynamics", namely the study of small perturbations of completely integrable systems. The complexity of such systems can be explained in terms of the geometric structures that organize the dynamics, that is, families of tori of KAM type and families of completely resonant tori, and their behavior under perturbations. Section 3 is on the averaged system or the secular system, which describes the long-term evolution of the shape and position of the ellipses osculating to the trajectories of the planets. Section 4 is devoted to the existence and continuation of periodic solutions, and on Poincaré's comments on the work of Hill on the lunar problem. It is followed by Section 5 on quasi-periodic solutions and on Lindstedt series.

The role of periodic solutions in establishing the non-integrability of the Three-Body Problem is the main point of Section 6. It also explains how the hyperbolic periodic orbits and their stable and unstable manifolds were introduced by Poincaré. Section 7 explains how Poincaré used Bohlin series to analyze simple resonances. The Poincaré recurrence theorem, its role as a precursor of ergodic theory, and its application to the Planar Circular Restricted Three-Body Problem (PCRTBP) are discussed in Section 8. The PCRTBP is then used in Section 9 to illustrate the method of the return map as a way to turn a continuous dynamical system into a discrete one. Still in the context of the PCRTBP, Section 10 describes Poincaré's argument that the stable and unstable manifolds of hyperbolic periodic orbits intersect at exponentially small angles, thus determining homoclinic tangles; this is reflected by the divergence of the corresponding Bohlin series. Section 11 explains the connections between Poincaré's work on quasi-periodic solutions and the later development of the KAM theorem. Section 12 discusses the last geometric theorem for twist diffeomorphisms of the annulus. The Principle of Least Action was put by Poincaré on the same footing as the great conservation principles; Section 13 looks at his work regarding this principle, as well as at the subsequent development pertaining to celestial mechanics. Section 14 discusses instability and diffusion in systems with more than two degrees of freedom. In this regard, Chenciner makes an interesting comment that, while there are many proofs of the Stability of (models of) the Solar System (in various conceptions of the notion of 'stability'), it seems quite possible (in the light of computer experiments) that the real Solar System is unstable (in every conceivable sense).

{For the collection containing this paper see MR3329471}

Marian Gidea